VII. *On the Steady Motion and Small Vibrations of a Hollow Vortex.*

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Contents.

The following pages form a continuation of some researches commenced about three years ago, but which the author was compelled by other engagements to lay aside until the beginning of the present year. The general theory of the functions employed was published in the Transactions of this Society (Part III., 1881), under the title of "Toroidal Functions." These and analogous functions are employed in the present communication, and references in square brackets, with the letters T.F., refer to this paper. Since it was written I have found that Carl Neumann had already given the general transformation [T.F. §1] by means of conjugate functions, in a pamphlet published at Halle in 1864, with the title 'Theorie der Elektricitäts- und Wärme-Vertheilung in einem Ringe.'

The theory of the motion of vortices is interesting, not only from the mathematical difficulties encountered in its treatment, but also from its connexion with Sir W. THOMSON's theory of the vortex atom constitution of matter. In an abstract of the present paper intended for the Proceedings of this Society, I have given some physical speculations which induced me to take up the question of the motion of a hollow vortex—that is, where cyclic motion exists in a fluid without the presence of any actual rotational filaments—in which case there must be a ring-shaped hollow in the fluid, however great the pressure may be, so long as it is finite. The essential quality of all vortex motion is the *cyclic* motion existing in the fluid outside the filament, and not the rotational motion of the filament itself. Whether the filament be present or not, it is often possible to get some general idea of the motion that ensues in many cases without recourse to actual calculation. Thus, for instance, the treatment by Sir W. THOMSON of the action of two vortices on one another,* and of the form of the axis of a ring, along which waves of displacement are running,† may be cited. The same course of general reasoning, which was applied in a paper on the steady motion of two cylinders in a fluid,‡ will also apply to illustrate the *mechanism*, so to speak, which causes a single vortex ring to move with a motion of translation. Thus suppose a single vortex ring, which is for a moment at rest. It is clear that the velocity of the fluid just inside the aperture is greater than outside, and therefore the pressure less inside than outside, whilst the pressure is the same at corresponding points in the front and hinder portions. The consequence of this is that the ring begins to contract without a general motion of translation. But the effect of this contraction of aperture itself produces velocities in the surrounding fluid, which, combined with the cyclic motion, increase the velocities in front of the ring, and decrease them behind. The consequence of this is a difference of pressures, which urges the ring in the direction of the cyclic motion through the ring, and it begins to move forward with increasing velocity. After a time this translatory motion would increase so much as to make the velocity within the aperture approach to that without; the state of motion will therefore be one in which the translatory velocity tends continually to a limit.

The present communication is divided into three sections. In the first, new functions are introduced to give the stream lines. These functions are connected with, and have analogous properties to, the Toroidal Functions; are, in fact, given by $R=SdP/du$ and $T=-SdQ/du$. They have the property of being single-valued, even when they represent cyclic motion—a motion which the single-valued Toroidal Functions cannot by themselves represent. At the end of the section the values of

* "Vortex Motion," Trans. Roy. Soc. Edin., xxv.

† "Vortex Statics," Proc. Roy. Soc. Edin., ix.

‡ Quart. Jour. Math., xvii.

the first few terms in the expansions of the first four orders of P, Q, R, T, are given. Section II. is devoted to the consideration of the motion of a rigid ring in fluid, when it moves parallel to its straight axis. The functions for the motion apply directly to the case considered afterwards of the vortex. The points of division of the stream, the quantity of fluid carried forward, and the energy of the motion are considered.

In Section III. the problem of the steady motion of a hollow vortex is treated, together with the small vibrations when the hollow is fluted, and when it pulsates. The section of a ring is throughout considered as small compared with the aperture, and the expressions giving the form of the hollow, the surface velocity, velocity of translation and energy, are carried to a second approximation, the quantity by which the approximation proceeds being the ratio $r/\{R+\surd(R^2-r^2)\}=k$ where r, $R-r$ denote the radii of the mean section and aperture respectively; when the ring is very small, this is very approximately $r/2R$. The condition that the hollow must be a free surface over which the pressure is constant gives a relation which R, r must always satisfy, which for very small rings reduces to the constancy of the radius of the hollow. For a solid ring the corresponding condition is, of course, the constancy of volume. This makes an essential difference between the two theories. To a second approximation the velocity of translation is unaltered, and is given by*

$$V=\frac{\mu}{4\pi a}\left(\log\frac{4}{k}-\tfrac{1}{2}\right)$$

whilst to the second approximation the surface velocity, relative to the hollow itself, is

$$U=\frac{\mu}{4\pi ak}\{1-\tfrac{1}{2}(\log\frac{4}{k}+5)k^2\}$$

where a is the radius of the "critical" circle—or the length of a tangent from the centre to the ring, and is therefore equal to R for small rings—and μ is the cyclic constant.

In the steady motion considered, the fluid carried forward with the ring forms a single mass, without aperture even for extremely small tores, though not for infinitely small ones. For values of $R/r>10^2$ there will be no aperture, whilst for less values the fluid carried forward will be ring-shaped. To a first approximation the energy due to the cyclic motion is the most important, and is the same as for a rigid ring at rest of the same size. It does not depend on the velocity of translation, except in so far as this determines the size of the aperture; as entering in this way the principal term varies inversely as the velocity of translation, and thus increases with diminished

* [April, 1884.—Owing to an error in § 3, the values given in the Proceedings require correction.]

translatory motion, a result obtained by Sir W. THOMSON* from general reasoning. The terms obtained by the second approximation arise from the translatory motion.

In Art. 13 the time of vibration of the steady form is obtained, when the cross section is crimped, or the whole hollow surface fluted. For this mode the time of vibration is, for small rings, given very approximately by $\mu d/(2p\sqrt{n})$, d being the density, and p the pressure of the fluid at a great distance, whilst n is the number of crimpings in a section. This, it is to be noted, is independent of the energy, and depends only on constants of the ring, and the fluid, and the mode of vibration. If the hollow pulsates, or changes its volume periodically, the time of pulsation is $(\mu d/2p)\sqrt{(\log 4/k)}$. As k depends on the size of the ring, and therefore on the energy, this time is not independent of the latter, but it varies extremely slowly with it. The times here given must be understood to apply to the steady motion; when the ring is changing its size they must be modified. The investigation of this case, and of that in which there is a core of denser matter than the surrounding fluid, I hope shortly to take up.

Section I.—*The functions.*

1. The functions whose properties were investigated in my paper on Toroidal Functions are only suitable for expressing fluid motions about circular tores when there is no cyclic motion through the aperture. It will be necessary therefore to investigate some method by which this can be taken into consideration. If we consider only motions symmetrical about an axis, and in planes through that axis, it is well known that the motion can be represented by STOKES' stream function. This function is only multiple valued when there are sources or sinks in the fluid, the cyclic constants in this case being the normal flows outwards through surfaces completely enclosing the various sources or sinks. If ψ denote the stream function, the velocities at any point are given by $-\frac{1}{\rho}\frac{\delta\psi}{\delta z}, \frac{1}{\rho}\frac{\delta\psi}{\delta\rho}$, and, when the motion is irrotational, ψ satisfies the equation

$$\frac{\delta^2\psi}{\delta\rho^2}+\frac{\delta^2\psi}{\delta z^2}-\frac{1}{\rho}\frac{\delta\psi}{\delta\rho}=0$$

To transform this to the independent variables $(u.v)$, where $u+vi=f(\rho+zi)$, we notice that the kinetic energy of fluid motion within any space, with given normal motions over this surface, is a minimum when the motion is irrotational, or the above differential equation is satisfied. The condition is therefore found by making

$$\int\frac{1}{\rho^2}\left\{\left(\frac{\delta\psi}{\delta u}\right)^2+\left(\frac{\delta\psi}{\delta v}\right)^2\right\}\left(\frac{du}{dn}\right)^2 2\pi\rho dz d\rho$$

a minimum. Now

* "Vortex Atoms," Proc. Roy. Soc. Edin., vi., and Phil. Mag. (4), 34.

$$dzd\rho=\frac{d(z.\rho)}{d(u.v)}dudv=\frac{1}{\left(\frac{du}{dn}\right)^2}dudv$$

Therefore the expression to be made a minimum is

$$\int\frac{1}{\rho}\left\{\left(\frac{\delta\psi}{\delta u}\right)^2+\left(\frac{\delta\psi}{\delta v}\right)^2\right\}dudv$$

whence

$$\frac{\delta}{\delta u}\left(\frac{1}{\rho}\frac{\delta\psi}{\delta u}\right)+\frac{\delta}{\delta v}\left(\frac{1}{\rho}\frac{\delta\psi}{\delta v}\right)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this put

$$\psi=\chi\sqrt{\rho}$$

and the equation in χ becomes, remembering that

$$\frac{\delta^2\rho}{\delta u^2}+\frac{\delta^2\rho}{\delta v^2}=0$$

$$\frac{\delta^2\chi}{\delta u^2}+\frac{\delta^2\chi}{\delta v^2}-\frac{3}{4}\frac{\chi}{\rho^2}\left\{\left(\frac{\delta\rho}{\delta u}\right)^2+\left(\frac{\delta\rho}{\delta v}\right)^2\right\}=0$$

The particular transformation employed for the Toroidal Functions makes

$$\frac{1}{\rho^2}\left\{\left(\frac{\delta\rho}{\delta u}\right)^2+\left(\frac{\delta\rho}{\delta v}\right)^2\right\}=\sinh^{-2}u=\mathrm{S}^{-2}$$

whence

$$\frac{\delta^2\chi}{\delta u^2}+\frac{\delta^2\chi}{\delta v^2}-\frac{3}{4\mathrm{S}^2}\chi=0$$

Put $\chi=\mathrm{S}^{-\frac{1}{2}}\mathrm{R}_n\cos(nv+\alpha)$, where R_n is a function of u only; then R_n must satisfy

$$\frac{d^2\mathrm{R}}{du^2}-\frac{\mathrm{C}}{\mathrm{S}}\frac{d\mathrm{R}}{du}-(n^2-\tfrac{1}{4})\mathrm{R}=0$$

which may be compared with the equation for Toroidal Functions, viz.,

$$\frac{d^2\mathrm{P}}{du^2}+\frac{\mathrm{C}}{\mathrm{S}}\frac{d\mathrm{P}}{du}-(n^2-\tfrac{1}{4})\mathrm{P}=0$$

It is easy to see that the equation in R is satisfied by

$$\mathrm{R}=\mathrm{AS}\frac{d\mathrm{P}}{du}$$

We will choose the constants so that the two integrals are

$$\left.\begin{aligned} R_n &= \quad S\frac{dP}{du}=\frac{4n^2-1}{8n}(P_{n+1}-P_{n-1}) \\ T_n &= -S\frac{dQ}{du}=\frac{4n^2-1}{8n}(Q_{n-1}-Q_{n+1}) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

also

$$\frac{dR_n}{du}=(n^2-\tfrac{1}{4})SP_n, \quad \frac{dT_n}{du}=-(n^2-\tfrac{1}{4})SQ_n$$

The value of ψ is now, putting in the value of ρ, viz., $\rho=aS/(C-c)$

$$\psi=\frac{1}{\sqrt{(C-c)}}\Sigma_0^\infty(A_nR_n+B_nT_n)\cos(nv+\alpha)$$

and clearly R, T belong to the same spaces as P, Q respectively, that is, R to space outside, and T to space inside a tore.

It is easy now to prove from the value of Q_n, viz.,

$$Q_n\sqrt{2}=\int_0^\pi\frac{\cos nv}{\sqrt{(C-c)}}dv$$

that

$$T_n=-\frac{4n^2-1}{2\sqrt{2}}\int_0^\pi\cos nv\sqrt{(C-c)}dv \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The R, T are all positive, except R_0.

2. *Cyclic constant.*—The cyclic constant of ψ is the flow along any closed curve threading the tore once. We know that this must be independent of the form of the curve. To find it, choose the curve to be $u=u'$ a constant; the flow along this is then

$$\int_0^{2\pi}\frac{1}{\rho}\frac{\delta\psi}{\delta u}\frac{du}{dn}\cdot\frac{dn}{dv}dv=\int_0^{2\pi}\frac{1}{\rho}\frac{\delta\psi}{\delta u}dv$$

the velocity in the aperture being in the positive direction. Consider first the general term in R_n; the flow due to this is

$$\frac{A_n}{aS}\int_0^{2\pi}\left\{\sqrt{(C-c)}\frac{dR_n}{du}-\frac{S}{2\sqrt{(C-c)}}R_n\right\}\cos nvdv$$

$$=\frac{2A_n}{aS}\int_0^\pi\left\{(n^2-\tfrac{1}{4})SP_n\sqrt{(C-\cos v)}\cos nv-\tfrac{1}{2}SR_n\frac{\cos nv}{\sqrt{(C-c)}}\right\}dv$$

$$=\frac{2A_n}{a}\left(-\frac{\sqrt{2}}{2}T_nP_n-\frac{\sqrt{2}}{2}R_nQ_n\right)$$

$$=\frac{A_n\sqrt{2}}{a}S\left(P_n\frac{dQ_n}{du}-Q_n\frac{dP_n}{du}\right)=-\frac{\pi}{a}\sqrt{2}A_n \quad [\text{T.F. } 24.\beta],$$

which is independent of u as it ought to be. The corresponding terms in sin nv evidently disappear. Similarly the terms in T_n would produce

$$-\frac{B_n\sqrt{2}}{a}S\left(Q_n\frac{dQ_n}{du}-Q_n\frac{dQ_n}{du}\right)=0$$

Hence the cyclic constant is

$$\mu=-\frac{\pi\sqrt{2}}{a}\Sigma_0^\infty A_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

3. In the paper on Toroidal Functions several examples were given of the determination of the potential function ϕ when ϕ is given over a tore; but when the variation of ϕ along the normal to the surface is given, the determination of the co-efficients becomes more difficult, and one case only, for the motion of a tore perpendicular to its plane, was given. It will be well, therefore, to consider here the general theory for this class of surface conditions. The co-efficients are to be determined from the fact that ϕ is (1) finite in the space considered, and at infinity, and (2) $d\phi/dn$ has a given value over the surface of a tore u'. Here I consider only the case where the motion is symmetrical about the axis, and therefore the normal velocity given by a function of v only, say $f(v)$. Condition (1) is satisfied by space outside the tore by taking only functions P_n. We put then

$$\phi=\sqrt{(C-c)}\Sigma_0(A_n\cos nv+B_n\sin nv)P_n$$

and determine A_n, B_n from the equation

$$f(v)=-\frac{\delta\phi}{\delta u}\cdot\frac{du}{dn}$$

when $u=u'$, for all values of v.

Consider separately the terms in cos nv and sin nv. For the cosines we have

$$\frac{\delta\phi}{\delta u}=\frac{1}{2\sqrt{(C-c)}}\Sigma_0A_n\left\{SP_n+2(C-c)\frac{dP_n}{du}\right\}\cos nv$$

For shortness write $\frac{dP}{du}=P'$. Then

$$\frac{\delta\phi}{\delta u}=\frac{1}{2\sqrt{(C-c)}}[(SP_0+2CP'_0)A_0-A_1P'_1-A_0P'_0\cos v^*+\Sigma_1\{(SP_n+2CP'_n)A_n$$
$$-A_{n+1}P'_{n+1}-A_{n-1}P'_{n-1}\}\cos nv]$$

But

$$SP_n+2CP'_n=P'_{n+1}+P'_{n-1} \quad \text{[T.F., p. 646]}$$
$$SP_0+2CP'_0=2P'_1$$

Therefore

$$\frac{\delta\phi}{\delta u}=\frac{1}{2\sqrt{(C-c)}}[(2A_0-A_1)P'_1-A_0P'_0\cos v+\Sigma_1\{(A_n-A_{n+1})P'_{n+1}-(A_{n-1}-A_n)P'_{n-1}\}\cos nv]$$

* [April, 1884.—This term was omitted in the paper as read. It has necessitated slight alterations in some of the results then given.]

and

$$\frac{du}{dn}=\frac{C-c}{a}$$

Hence the A have to be determined from

$$\frac{2a}{(C-c)^{\frac{1}{2}}}f(v)=-(2A_0-A_1)P'_1+A_0P'_0\cos v-\Sigma_1\{(A_n-A_{n+1})P'_{n+1}-(A_{n-1}-A_n)P'_{n-1}\}\cos nv$$

Suppose now

$$f(v)=\left(\frac{C-c}{S}\right)^{\frac{1}{2}}\alpha_m\cos mv$$

Then, writing for the present $(A_n-A_{n-1})P'_{n-1}P'_n=x_n$

$$\left.\begin{array}{r} x_{n+1}-x_n=0 \\ \ldots\ldots=0 \end{array}\right\} \quad n>m$$

$$x_{m+1}-x_m=\frac{2a\alpha_m}{\sqrt{S}}P'_m$$

$$\left.\begin{array}{r} x_{n+1}-x_n=0 \\ \ldots\ldots=0 \\ x_2-x_1+A_0P'_0P'_1=0 \\ x_1-x_0=0 \end{array}\right\} \quad n<m \quad \text{where } x_0=A_0P'_0P'_1$$

Hence for

$$n>m+1 \qquad x_n=x_{m+1}$$
$$n<m \qquad x_m=x_n=0$$

and

$$x_{m+1}=\frac{2a\alpha_m}{\sqrt{S}}P'_m$$

or

$$x_n=\frac{2a\alpha_m}{\sqrt{S}}P'_m \quad . \quad (n>m)$$
$$x_n=0 \qquad (n\overline{\gtrless}m)$$

Therefore

$$A_n-A_{n-1}=\frac{2a\alpha_m}{\sqrt{S}}\cdot\frac{P'_m}{P'_nP'_{n-1}} \quad (n>m)$$
$$A_n-A_{n-1}=0 \qquad (n\overline{\gtrless}m>1)$$

whence

$$A_n=A_m+\frac{2a\alpha_m}{\sqrt{S}}P'_m\sum_{r=m+1}^{r=n}\frac{1}{P'_rP'_{r-1}} \quad (n>m)$$
$$A_n=A_1=2A_0 \qquad (n\overline{\gtrless}m)$$

or

$$A_n = 2A_0 + \frac{2a\alpha_m}{\sqrt{S}} P'_m \sum_{r=m+1}^{r=n} \frac{1}{P'_r P'_{r-1}} \qquad (n>m)$$

$$A_n = 2A_0 \qquad (n \overline{\leq} m)$$

The co-efficients are now determined to the extent of one arbitrary constant. This appears because ϕ is also indeterminate to the extent of an additive constant. As this constant is expansible in a series of the form $\sqrt{(C-c)}\Sigma A_n \cos nv$, it introduces the undetermined constant A_0 above, which must be determined by the condition that the series must be convergent. This cannot be unless $A_\infty = 0$, which requires

$$A_0 = -\frac{a\alpha_m P'_m}{\sqrt{S}} \Sigma_{m+1}^{\infty} \frac{1}{P'_r P'_{r-1}}$$

whence

$$\left.\begin{aligned} A_n &= -\frac{2a\alpha_m P'_m}{\sqrt{S}} \Sigma_{n+1}^{\infty} \frac{1}{P'_r P'_{r-1}} \qquad (n>m) \\ A_n &= -\frac{2a\alpha_m P'_m}{\sqrt{S}} \Sigma_{m+1}^{\infty} \frac{1}{P'_r P'_{r-1}} \qquad (n \overline{\leq} m) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

So also the terms in sin nv will give

$$\frac{2a}{(C-c)^{\frac{1}{2}}} f(v) = -\Sigma_1 \{(B_n - B_{n+1})P'_{n+1} - (B_{n-1} - B_n)P'_{n-1}\} \sin nv$$

and the particular case $f(v) = \left(\frac{C-c}{S}\right)^{\frac{1}{2}} \beta_m \sin mv$ produces the same equations as before, except that the last is

$$x_2 - x_1 = 0 \text{ where } x_1 = B_1 P'_0 P'_1$$

whence

$$B_n = B_1 P'_0 P'_1 \Sigma_1^n \frac{1}{P'_r P'_{r-1}} + \frac{2a\beta_m}{\sqrt{S}} P'_m \Sigma_{m+1}^n \frac{1}{P'_r P'_{r-1}} \qquad (n>m)$$

$$B_n = B_1 P'_0 P'_1 \Sigma_1^n \frac{1}{P'_r P'_{r-1}} \qquad (n \overline{\leq} m)$$

and the condition of convergency determines B_1, so that

$$\left.\begin{aligned} B_n &= -\frac{2a\beta_m P'_m}{\sqrt{S}} \frac{\Sigma_1^m \frac{1}{P'_r P'_{r-1}}}{\Sigma_1^\infty \frac{1}{P'_r P'_{r-1}}} \cdot \Sigma_{n+1}^\infty \frac{1}{P'_r P'_{r-1}} \qquad (n > m) \\ B_n &= -\frac{2a\beta_m P'_m}{\sqrt{S}} \cdot \frac{\Sigma_{m+1}^\infty \frac{1}{P'_r P'_{r-1}}}{\Sigma_1^\infty \frac{1}{P'_r P'_{r-1}}} \Sigma_1^n \frac{1}{P'_r P'_{r-1}} \qquad (n \leqq m) \end{aligned}\right\} \quad \ldots \ldots \quad (6)$$

It remains to show that with these values of A_n, B_n the series $\Sigma A_n P_n$ and $\Sigma B_n P_n$ are convergent. The parts of A_n, B_n depending on n, when n is large,

$$\propto \Sigma_{n+1}^\infty \frac{1}{P'_r P'_{r-1}} = \lambda \Sigma_{n+1}^\infty \frac{1}{P'_r P'_{r-1}}$$

Now

$$\frac{P'_{n-1}}{P'_n} = \frac{P_n - CP_{n-1}}{CP_n - P_{n-1}} = \frac{1}{C} \cdot \frac{P_n - CP_{n-1}}{P_n - P_{n-1}/C} \qquad [\text{T.F., 12, 13}]$$

$$< \frac{1}{C} \text{ since } P_n > CP_{n-1}$$

Hence

$$\Sigma_{n+1}^\infty \frac{1}{P'_r P'_{r-1}} < \frac{1}{P'_{n+1} P'_{n-1}} \left\{ 1 + \frac{1}{C} + \frac{1}{C^2} + \ldots \right\}$$

$$< \frac{C}{C-1} \frac{1}{P'_{n+1} P'_n}$$

Therefore

$$\left.\begin{matrix} A_n \\ B_n \end{matrix}\right| P_n \text{ is ultimately } < \frac{\lambda C}{C-1} \cdot \frac{P_n}{P'_{n+1} P'_n}$$

or the series are convergent.

We are now in a position to determine ϕ for any normal motion. All we have to do is to expand $\{S/(C-c)\}^{\frac{1}{2}} f(v)$ in a series of sines and cosines of multiples of v and consider each term as giving rise to a value of ϕ, whose form we have just determined, and take the sum of the various values.

A form for $\delta\psi/\delta u$ analogous to that for $\delta\phi/\delta u$ can easily be found. If

$$\psi = \frac{1}{\sqrt{(C-c)}} \Sigma A_n R_n \cos nv$$

$$\frac{\delta\psi}{\delta u} = \frac{1}{(C-c)^{\frac{3}{2}}} \Sigma \left\{ (C-c) \frac{dR_n}{du} - \tfrac{1}{2} SR_n \right\} A_n \cos nv$$

$$= \frac{1}{2} \frac{1}{(C-c)^{\frac{3}{2}}} \Sigma \left\{ (2C \frac{dR_n}{du} - SR_n) \cos nv - (\cos \overline{n+1}v + \cos \overline{n-1}v) \frac{dR_n}{du} \right\} A_n$$

$$= \frac{S}{2(C-c)^{\frac{3}{2}}} \left[-\tfrac{1}{2} A_0 P_1 - \tfrac{3}{4} A_1 P_1 + \frac{1}{S} \Sigma_1 \left\{ (2C \frac{dR_n}{du} - SR_n) A_n - A_{n+1} \frac{dR_{n+1}}{du} - A_{n-1} \frac{dR_{n-1}}{du} \right\} \cos nv \right]$$

But

$$2C\frac{dR_n}{du}-SR_n=(n^2-\tfrac{1}{4})\left\{2CSP_n-\frac{S}{2n}(P_{n+1}-P_{n-1})\right\}=(n^2-\tfrac{1}{4})S(P_{n+1}+P_{n-1})$$

Hence

$$\frac{\delta\psi}{\delta u}=\frac{S}{2(C-c)^{\frac{3}{2}}}\left\{-\tfrac{1}{2}A_0P_1-\tfrac{3}{4}A_1P_1+\tfrac{1}{4}A_0P_0\cos v+\Sigma_1 B_n\cos nv\right\} \quad . \quad . \quad . \quad (7)$$

where

$$B_n=\{(n^2-\tfrac{1}{4})A_n-(\overline{n+1}|^2-\tfrac{1}{4})A_{n+1}\}P_{n+1}-\{(\overline{n-1}|^2-\tfrac{1}{4})A_{n-1}-(n^2-\tfrac{1}{4})A_n\}P_{n-1}$$

4. For reference I here insert the values of the first five orders of the functions, expressed (α) exact in terms of the elliptic integrals, and (β) approximate in a series of ascending powers of the modulus. Throughout this paper the moduli k, k' are used instead of the k', k of the paper on Toroidal Functions. It has been thought advisable to do this as all the approximations go according to powers of k (the old k'). Hence, of course, E, F appear in place of E′, F′, and *vice versâ*.

We know that $P_n=\alpha_n E'+\beta_n F'$, $Q_n=\alpha_n(F-E)+\beta_n F$.

Hence for the first set of formulæ we require only to tabulate α_n, β_n. For the first five they are

$$\left.\begin{array}{ll}
\alpha_0=0 & \beta_0=2k^{\frac{1}{2}} \\
\alpha_1=2k^{-\frac{1}{2}} & \beta_1=0 \\
\alpha_2=\tfrac{4}{3}(1+k^2)k^{-\frac{3}{2}} & \beta_2=-\tfrac{2}{3}k^{\frac{1}{2}} \\
\alpha_3=\tfrac{2}{15}(8+7k^2+8k^4)k^{-\frac{5}{2}} & \beta_3=-\tfrac{8}{15}(k^2+k^4)k^{-\frac{5}{2}} \\
\alpha_4=\dfrac{16}{3.5.7}(6+5k^2+5k^4+6k^6)k^{-\frac{7}{2}}, & \beta_4=-\dfrac{2}{3.5.7}(24k^2+23k^4+24k^6)k^{-\frac{7}{2}} \\
\alpha_5=\dfrac{2}{5.7.9}(128+104k^2+99k^4+104k^6+128k^8)k^{-\frac{9}{2}}, & \beta_5=-\dfrac{8}{5.7.9}(16k^2+15k^4+15k^6 \\
 & \qquad +16k^8)k^{-\frac{9}{2}}
\end{array}\right\} \quad (8)$$

These are exact. In the applications which follow k will nearly always be a very small quantity, so that a few terms of the series will give the values very approximately. By substituting their values for E, F, E′, F′ in terms of k, the expressions become, writing L for $\log\frac{4}{k}$

$$\left.\begin{array}{l}
P_0=2\{L+\tfrac{1}{4}(L-1)k^2+\tfrac{9}{64}(L-\tfrac{7}{6})k^4+\tfrac{25}{256}(L-\tfrac{37}{30})k^6+\ldots\}k^{\frac{1}{2}} \\
P_1=2\{1+\tfrac{1}{2}(L-\tfrac{1}{2})k^2+\tfrac{3}{16}(L-\tfrac{13}{12})k^4+\tfrac{15}{128}(L-\tfrac{6}{5})k^6+\ldots\}k^{-\frac{1}{2}} \\
P_2=\tfrac{4}{3}\{1+\tfrac{3}{4}k^2+\tfrac{9}{16}(L-\tfrac{7}{12})k^4+\tfrac{15}{64}(L-\tfrac{67}{20})k^6+\ldots\}k^{-\frac{3}{2}} \\
P_3=\tfrac{16}{15}\{1+\tfrac{5}{8}k^2+\tfrac{45}{64}k^4+\tfrac{75}{128}(L-\tfrac{37}{60})k^6+\ldots\}k^{-\frac{5}{2}} \\
P_4=\tfrac{32}{35}\{1+\tfrac{7}{12}k^2+\tfrac{35}{64}k^4+\tfrac{175}{256}k^6+\ldots\}k^{-\frac{7}{2}} \\
P_5=\dfrac{256}{5.7.9}\{1+\tfrac{9}{16}k^2+\tfrac{63}{128}k^4+\tfrac{525}{1024}k^6+\ldots\}k^{-\frac{9}{2}}
\end{array}\right\} \quad . \quad . \quad (9)$$

$$\left.\begin{aligned}
Q_0 &= \pi\{1+\tfrac{1}{4}k^2+\tfrac{9}{64}k^4+\tfrac{25}{256}k^6+\ldots\}k^{\frac{1}{2}}\\
Q_1 &= \tfrac{1}{2}\pi\{1+\tfrac{3}{8}k^2+\tfrac{15}{64}k^4+\ldots\}k^{\frac{3}{2}}\\
Q_2 &= \frac{3\pi}{8}(1+\tfrac{5}{12}k^2+\ldots)k^{\frac{5}{2}}\\
Q_3 &= \frac{5\pi}{16}k^{\frac{7}{2}}+\ldots\\
Q_4 &= 0+\ldots\\
Q_5 &= 0+\ldots
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (10)$$

$$\left.\begin{aligned}
R_0 &= -\{\tfrac{1}{2}L-1+\tfrac{1}{8}(L+1)k^2+\tfrac{1}{128}(L-\tfrac{1}{2})k^4+\tfrac{1}{512}(L-\tfrac{5}{6})k^6+\ldots\}k^{-\frac{1}{2}}\\
R_1 &= \tfrac{1}{2}\{1-\tfrac{3}{2}(L-\tfrac{1}{2})k^2+\tfrac{3}{16}(L+\tfrac{1}{4})k^4+\tfrac{3}{128}(L-23)k^6+\ldots\}k^{-\frac{3}{2}}\\
R_2 &= \{1-\tfrac{5}{4}k^2-\tfrac{15}{16}(L-\tfrac{5}{4})k^4+\tfrac{15}{64}(L+\tfrac{1}{12})k^6+\ldots\}k^{-\frac{5}{2}}\\
R_3 &= \tfrac{4}{3}\{1-\tfrac{7}{8}k^2-\tfrac{35}{64}k^4-\tfrac{35}{32}(3L-\tfrac{17}{4})k^6+\ldots\}k^{-\frac{7}{2}}\\
R_4 &= \tfrac{8}{5}\{1-\tfrac{3}{4}k^2-\tfrac{21}{64}k^4-\tfrac{105}{256}k^6+\ldots\}k^{-\frac{9}{2}}
\end{aligned}\right\} \quad . \quad (11)$$

$$\left.\begin{aligned}
T_0 &= \tfrac{1}{4}\pi(1+\tfrac{1}{4}k^2+\tfrac{1}{64}k^4+\tfrac{1}{256}k^6+\ldots)k^{-\frac{1}{2}}\\
T_1 &= \frac{3\pi}{8}(1-\tfrac{1}{8}k^2-\tfrac{1}{64}k^4+\ldots)k^{\frac{1}{2}}\\
T_2 &= \frac{15\pi}{32}(1-\tfrac{1}{4}k^2+\ldots)k^{\frac{3}{2}}\\
T_3 &= \tfrac{35}{64}\pi k^{\frac{5}{2}}+\ldots
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (12)$$

Section II.—*Motion about a rigid tore which moves perpendicularly to its plane.*

As the motion of a tore throws some light on the analogous problem of the uniform translation of a vortex ring, and as the functions required in its discussion will be needed in investigating the latter, it will be useful to give a short treatment of the question, especially as the motion can be determined for any size of tore, whereas our methods, in the case of hollow vortices, will only apply when the cross section of the hollow is not large compared with the aperture. The stream function is necessary for the cyclic motion, and it will therefore be convenient to take the stream function also for the motion of translation.

5. *Stream function for cyclic motion.*—If the tore be given by $u=u'$ the conditions which ψ must satisfy are that it must be finite for space outside the tore, and be constant for all values of v when $u=u'$. Hence ψ must be expansible in the form

$$\psi=\frac{1}{\sqrt{(C-c)}}\Sigma A_n R_n \cos nv$$

Let ψ_0 be the constant value over the surface of the tore. Then, dashed letters denoting the values of the functions on the tore,

$$\frac{\pi}{2}\,\mathrm{A}_n\mathrm{R}'_n=\psi_0\int_0^\pi\sqrt{(\mathrm{C}'-\cos\theta)}\cos n\theta d\theta$$

$$=-\frac{2\sqrt{2}\psi_0}{4n^2-1}\mathrm{T}'_n \qquad \text{(by Eq. 3)}$$

but

$$\pi\mathrm{A}_0\mathrm{R}'_0=2\psi_0\sqrt{2}\mathrm{T}'_0$$

Hence

$$\psi=\frac{2\psi_0\sqrt{2}}{\pi\sqrt{(\mathrm{C}-c)}}\left\{\mathrm{T}'_0\frac{\mathrm{R}_0}{\mathrm{R}'_0}-2\Sigma_1^\infty\frac{\mathrm{T}'_n}{4n^2-1}\frac{\mathrm{R}_n}{\mathrm{R}'_n}\cos nv\right\} \quad \ldots \ldots \quad (13)$$

This is more convergent than $\Sigma\mathrm{T}_n$, and is therefore convergent.

Let μ denote the cyclic constant, then by (4)

$$\mu=\frac{4\psi_0}{a}\left\{-\frac{\mathrm{T}'_0}{\mathrm{R}'_0}+2\Sigma_1^\infty\frac{1}{4n^2-1}\cdot\frac{\mathrm{T}'_n}{\mathrm{R}'_n}\right\} \quad \ldots \ldots \quad (14)$$

When the section of the tore is small compared with the aperture, the value of μ, correct to the fourth power of k, is

$$\mu=\frac{2\pi\psi_0}{a}\left\{\frac{1}{\mathrm{L}-2}+\left(1-\frac{3}{4}\frac{1}{(\mathrm{L}-2)^2}\right)k^2+\frac{3}{4}\left(2\mathrm{L}-1+\frac{1}{32}\frac{7\mathrm{L}+10}{(\mathrm{L}-2)^3}\right)k^4\right\} \quad \ldots \quad (15)$$

6. *Stream function for translation.*—In the preceding case the conditions were that ψ must be constant over the tore and finite at an infinite distance from it. In the present case ψ must be finite at an infinite distance and $=\frac{1}{2}\mathrm{V}\rho^2$ over the surface, V being the velocity of translation, and ψ the stream function for the tore moving in the fluid, at rest at infinity and referred to its instantaneous position. But if this condition be applied, we shall also, on account of the cyclosis, obtain besides an added cyclic motion through the aperture determined by the surface condition $\psi_0=0$. It will be necessary to subtract this cyclic motion therefore from the result obtained by applying the condition above. This condition gives

$$\Sigma\mathrm{A}_n\mathrm{R}'_n\cos nv=\frac{1}{2}a^2\mathrm{V}\frac{\mathrm{S}'^2}{(\mathrm{C}'-c)^{\frac{3}{2}}}$$

for all values of v.

Therefore

$$\tfrac{1}{2}\pi A_n R'_n = \tfrac{1}{2}a^2 V S'^2 \int_0^\pi \frac{\cos n\theta}{(C' - \cos\theta)^{\frac{3}{2}}}\,d\theta$$

$$= -a^2 V S' \frac{d}{du}\int_0^\pi \frac{\cos n\theta}{\sqrt{C' - \cos\theta}}\,d\theta$$

$$= -a^2 V \sqrt{2} S' \frac{dQ'}{du} = a^2 V \sqrt{2} T'_n$$

but

$$\pi A_0 R'_0 = a^2 V \sqrt{2} T'_0$$

Hence

$$\psi = \frac{a^2\sqrt{2}V}{\pi\sqrt{(C-c)}}\left\{T'_0\frac{R_0}{R'_0} + 2\Sigma_1^\infty T'_n \frac{R_n}{R'_n}\cos nv\right\}$$

a convergent series.

The circulation of this is by (4)

$$2aV\left(-\frac{T'_0}{R'_0} - 2\Sigma_1^\infty \frac{T'_n}{R'_n}\right)$$

Let the stream function for this be

$$\frac{2\psi_0\sqrt{2}}{\pi\sqrt{(C-c)}}\left\{T'_0\frac{R_0}{R'_0} - 2\Sigma_1^\infty \frac{T'_n}{4n^2-1}\frac{R_n}{R'_n}\cos nv\right\}$$

Then

$$\frac{4\psi_0}{a}\left\{-\frac{T'_0}{R'_0} + 2\Sigma_1^\infty \frac{1}{4n^2-1}\frac{T'_n}{R'_n}\right\} = 2aV\left(-\frac{T'_0}{R'_0} - 2\Sigma_1^\infty \frac{T'_n}{R'_n}\right)$$

If then

$$\lambda = \frac{-\dfrac{T'_0}{R'_0} - 2\Sigma_1^\infty \dfrac{T'_n}{R'_n}}{-\dfrac{T'_0}{R'_0} + 2\Sigma_1^\infty \dfrac{1}{4n^2-1}\dfrac{T'_n}{R'_n}}$$

The stream function for translation alone is

$$\psi = \frac{a^2\sqrt{2V}}{\pi\sqrt{(C-c)}}\left\{(1-\lambda)T'_0\frac{R_0}{R'_0} + 2\Sigma_1^\infty\left(1 + \frac{\lambda}{4n^2-1}\right)T'_n\frac{R_n}{R'_n}\cos nv\right\} \quad . \quad . \quad . \quad (16)$$

The principal term here is the second, in R_1. To k^4 the value of λ is

$$\lambda = 1 - 4(L-2)k^2 - \left(2L^2 - \frac{21}{2}L + 16\right)k^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The value of ψ along the tore is

$$\tfrac{1}{2}V(\rho^2 - \lambda a^2)$$

The stream lines will of course in general be closed curves, having their extremities on the surface of the tore; one set going through the aperture, and the other outside.

To find the point where the two sets meet on the tore, we notice that the stream line there goes to infinity, and its value is the same as for a point on the axis, it is, in fact, a part of the same stream line. For this $\psi=0$; hence the point on the tore, where this stream line meets it, is given by the value of v, which satisfies the equation

$$(1-\lambda)T_0+2\Sigma_1^\infty\left(1+\frac{\lambda}{4n^2-1}\right)T_n \cos nv=0$$

where T_n are the values of T_n when $u=u'$.

It is clear that when k is very small, $\cos v$ must be negative, that is $v>\frac{1}{2}\pi$, or that the point of division must lie inside a tangent from the centre to the tore.

7. *Combined translation and cyclic motion.*—The expressions just obtained enable us to determine the amount of fluid carried forward bodily with the ring. Let x denote the ratio $a^2V/2\psi_0$; then the stream function for the combined motion is

$$\psi=\frac{2\psi_0\sqrt{2}}{\pi\sqrt{(C-c)}}\Sigma A_n\frac{R_n}{R'_n}\cos nv$$

where

$$A_0=\{1+(1-\lambda)x\}T'_0$$

$$A_n=2\left\{\left(1+\frac{\lambda}{4n^2-1}\right)x-\frac{1}{4n^2-1}\right\}T'_n.$$

This is the stream function when the fluid is at rest at an infinite distance. To find the portion carried forward, impress on the whole system a velocity equal and opposite to V; the problem then is to determine the portion of fluid which remains circulating round the ring at rest, without streaming away. The stream function for the new motion is

$$\chi=\psi-\tfrac{1}{2}V\rho^2$$

The portion remaining with the tore lies inside the surface given by putting χ equal to a certain constant, which we proceed to determine.

This portion may either be ring-shaped or not. The limiting case between the two is when the velocity at the centre of the tore is zero. The value of x for this case we shall call the critical value of x. It is given by

$$V=\left[\frac{1}{\rho}\frac{\delta\psi}{\delta u}\frac{du}{dn}\right]_{u=0,v=\pi}$$

$$=\frac{8\psi_0\sqrt{2}}{\pi a^2}L_{u=0}\frac{1}{S}\left\{-\frac{1}{2}\frac{S}{2^{\frac{3}{2}}}\Sigma(-)^n A_n\frac{R_n}{R'_n}+\frac{1}{\sqrt{2}}\Sigma(-)^n A_n(n^2-\tfrac{1}{4})\frac{SP_n}{R'_n}\right\}$$

$$=\frac{8\psi_0}{a^2}\Sigma(-)^n(n^2-\tfrac{1}{4})\frac{A_n}{R'_n}$$

or

$$x = -\{1+(1-\lambda)x\}\frac{T'_0}{R'_0} + 2\Sigma_1(-)^n\{(4n^2-1+\lambda)x-1\}\frac{T'_n}{R'_n}$$

$$x\left\{1+(1-\lambda)\frac{T'_0}{R'_0}+2\Sigma_1(-)^{n-1}(4n^2-1+\lambda)\frac{T'_n}{R'_n}\right\} = -\frac{T'_0}{R'_0}+2\Sigma_1(-)^{n-1}\frac{T'_n}{R'_n}$$

The right hand member of this equation is the velocity at the centre due to the cyclic motion alone, divided by $2\psi_0/a^2$. Call this velocity V_1, and denote the critical value of V by V_0, then

$$\frac{V_1}{V_0} = 1+(1-\lambda)\frac{T'_0}{R'_0}+2\Sigma_1(-)^{n-1}(4n^2-1+\lambda)\frac{T'_n}{R'_n}$$

The most important terms in these expressions are

$$\left.\begin{aligned} x_0 &= \frac{\pi}{2(L-2)}\left\{1+\left(3L-6-4\pi-\frac{3}{4}\frac{1}{L-2}\right)k^2+\ldots\right\} \\ \frac{V_1}{V_0} &= 1+4\pi k^2+\ldots \end{aligned}\right\} \qquad (18)$$

The stream lines will be given by

$$\psi - \tfrac{1}{2}V\rho^2 = \text{const}$$

and by choosing the constant properly, we may make this represent the surface of the fluid carried forward. To determine the constant we need only find one point on the surface by the above method. If the value of x is less than the critical value, the surface will extend to the axis; in this case the best way will be to put $u=0$ and find v from the equation

$$V = \left[\frac{1}{\rho}\frac{\delta\psi}{du}\frac{du}{dn}\right]_{u=0}$$

If on the contrary x is greater than x_0, the surface is ring-shaped, and it will be best to find u from the equation

$$V = \left[\frac{1}{\rho}\frac{\delta\psi}{du}\frac{du}{dn}\right]_{v=\pi}$$

If x be negative, or the velocity of translation and the cyclic motion within the aperture be in the opposite direction, the corresponding equation will be

$$V = \left[\frac{1}{\rho}\frac{d\psi}{du}\cdot\frac{du}{dn}\right]_{v=0}$$

In tabulating corresponding values of u, v and V/V_0 the best way would be to insert values of u, v and determine V/V_0. The following numbers in the case of $k=\sin 1°$ were obtained in this way. For the case of x less than the critical value, the surface cuts the straight axis at points given in the several cases by v,

v	60°	90°	120°	180°
$\frac{V}{V_0}$	·125	·355	·652	1

For a ring-shaped surface

$$u=2\cdot9662,\ v=180,\ \text{and}\ V/V_0=1\cdot699$$

whilst for a negative translation

$$u=2\cdot9662,\ v=0,\ \text{and}\ V/V_0=-\cdot3708$$

are sets of corresponding values.

8. *The energy of the fluid motion.*—The energy is given by

$$E=\tfrac{1}{2}\iint\frac{1}{\rho^2}\left\{\left(\frac{\delta\psi}{\delta\rho}\right)^2+\left(\frac{\delta\psi}{\delta z}\right)^2\right\}2\pi\rho d\rho dz=\pi\iint\frac{1}{\rho}\left\{\left(\frac{\delta\psi}{\delta\rho}\right)^2+\left(\frac{\delta\psi}{\delta z}\right)^2\right\}d\rho dz$$

supposing the density of the fluid to be unity. Treating this in a similar way to the analogous expression in terms of the velocity potential, and remembering that whenever the volume of the surfaces immersed remains constant, as here, ψ is single valued, we shall find (by means of equation 1) that

$$E=\pi\int\frac{1}{\rho}\psi\frac{\delta\psi}{\delta n}ds$$

the integration being extended over any meridian curve of the solid, and dn being measured inwards along the normal (*i.e.*, from the fluid).

In the case, therefore, of circular tores

$$E=\pi\int_0^{2\pi}\frac{1}{\rho}\psi\frac{\delta\psi}{\delta u}\cdot\frac{\delta u}{\delta n}\frac{dn'}{dv}.dv=\pi\int_0^{2\pi}\frac{\psi}{\rho}\frac{\delta\psi}{\delta u}dv$$

Now we know that for the cyclic motion the energy is $\frac{1}{2}\times$ cyclic constant $\times$ flow through the aperture, and, therefore, with our notation is $\mu\times\pi\psi_0$. But it will be interesting to see how this is also arrived at from the preceding expression. The whole energy can be put in the form

$$E=(\alpha\mu^2+\beta x^2+2\gamma\mu x)\psi_0^2$$

we proceed to determine α, β, γ by means of the above formula.

α. Here along the surface $\psi=\psi_0$ a constant, and,

$$\frac{1}{\rho}\frac{\delta\psi}{\delta u}=\frac{1}{a}\Sigma\left\{-\frac{\mathrm{R}_n}{2\sqrt{(\mathrm{C}-c)}}+\frac{4n^2-1}{4}\mathrm{P}_n\sqrt{(\mathrm{C}-c)}\right\}\mathrm{A}_n\cos nv$$

Therefore

$$\int_0^{2\pi}\frac{1}{\rho}\frac{\delta\psi}{\delta u}dv=\frac{2}{a}\Sigma\left\{-\tfrac{1}{2}\mathrm{R}_n\mathrm{Q}_n\sqrt{2}-\frac{1}{\sqrt{2}}\mathrm{P}_n\mathrm{T}_n\right\}\mathrm{A}_n$$

$$=-\frac{\sqrt{2}}{a}\Sigma\mathrm{S}\left(\mathrm{Q}_n\frac{d\mathrm{P}_n}{du}-\mathrm{P}_n\frac{d\mathrm{Q}_n}{du}\right)\mathrm{A}_n$$

$$=-\frac{\pi\sqrt{2}}{a}\Sigma\mathrm{A}_n \qquad [\text{T.F., 24}]$$

$$=\mu \qquad (\text{by } 4)$$

Hence energy $=\pi\mu\psi_0$, as is right, or

$$\alpha=\frac{\pi}{\mu\psi_0}=\frac{\pi a}{4\psi_0^2}\frac{1}{-\frac{\mathrm{T}_0}{\mathrm{R}_0}+2\Sigma_1^\infty\frac{1}{4n^2-1}\frac{\mathrm{T}_n}{\mathrm{R}_n}}$$

β. Here ψ along the tore is $\frac{1}{2}\mathrm{V}(\rho^2-\lambda a^2)$, and denoting the general value of ψ by $\mu\psi_1+x\psi_2$

$$\beta\psi_0^2x^2=\pi x\int_0^{2\pi}\tfrac{1}{2}\mathrm{V}(\rho^2-\lambda a^2)\frac{1}{\rho}\frac{\delta\psi_2}{\delta u}dv$$

Now $\int_0^{2\pi}\frac{1}{\rho}\frac{\delta\psi_2}{du}dv$ is proportional to the flow round the ring due to translation alone, and is therefore zero.

Hence

$$\beta=\frac{\pi}{a^2\psi_0}\int_0^{2\pi}\rho\frac{\delta\psi_2}{\delta u}dv$$

But

$$\psi_2=\frac{2\psi_0\sqrt{2}}{\pi\sqrt{(\mathrm{C}-c)}}\left\{(1-\lambda)\mathrm{T}'_0.\frac{\mathrm{R}_0}{\mathrm{R}'_0}+2\Sigma\left(1+\frac{\lambda}{4n^2-1}\right)\mathrm{T}'_n\frac{\mathrm{R}_n}{\mathrm{R}'_n}\cos nv\right\}$$

$$=\frac{2\psi_0\sqrt{2}}{\pi\sqrt{(\mathrm{C}-c)}}\Sigma\mathrm{A}_n\mathrm{R}_n\cos nv \qquad (\text{say})$$

Therefore

$$\beta=\frac{2\mathrm{S}'\sqrt{2}}{a}\int_0^{2\pi}\frac{1}{\mathrm{C}'-c}.\frac{\delta\psi_2}{\delta u}dv$$

Hence by 7

$$\beta=\frac{4\mathrm{S}'\sqrt{2}}{a}\int_0^{\pi}\frac{1}{2}\frac{\mathrm{S}'}{(\mathrm{C}-c)^{\frac{5}{2}}}[-\tfrac{1}{2}\mathrm{A}_0\mathrm{P}_1-\tfrac{3}{4}\mathrm{A}_1\mathrm{P}_1+\tfrac{1}{4}\mathrm{A}_0\mathrm{P}_0\cos v+\Sigma_1\mathrm{B}_n\cos nv]dv$$

Now

$$\mathrm{Q}_n\sqrt{2}=\int_0^{\pi}\frac{\cos nv dv}{(\mathrm{C}-c)^{\frac{1}{2}}}$$

Therefore

$$\int_0^\pi \frac{\cos nv}{(C-c)^{\frac{3}{2}}}dv = -\frac{2\sqrt{2}}{S}\frac{dQ_n}{du}$$

$$\int_0^\pi \frac{\cos nv}{(C-c)^{\frac{5}{2}}} = \frac{4\sqrt{2}}{3S}\frac{d}{du}\left(\frac{1}{S}\frac{dQ_n}{du}\right)$$

$$S^2\int_0^\pi \frac{\cos nv}{(C-c)^{\frac{5}{2}}} = \frac{4\sqrt{2}}{3}\left(\frac{d^2Q_n}{du^2}-\frac{C}{S}\frac{dQ_n}{du}\right)$$

$$= \frac{\sqrt{2}}{3}\left\{(4n^2-1)Q_n-\frac{8C}{S}\frac{dQ_n}{du}\right\}$$

Hence dropping the dashes, and u denoting the value of u along the tore

$$\beta=\frac{4}{3a}\left[\tfrac{1}{2}(A_0+\tfrac{3}{2}A_1)P_1\left(Q_0-\frac{8C}{S^2}T_0\right)+\tfrac{1}{4}A_0P_0\left(3Q_1+\frac{8C}{S^2}T_1\right)+\Sigma_1 B_n\left\{(4n^2-1)Q_n+\frac{8C}{S^2}T_n\right\}\right]$$

A_n having the values given above, and B_n the values given in 7.

γ. The value of γ is given by

$$2\gamma\mu x\psi_0{}^2=\pi\int_0^{2\pi}\frac{\mu x}{\rho}\left(\psi_1\frac{\delta\psi_2}{\delta u}+\psi_2\frac{\delta\psi_1}{\delta u}\right)dv$$

Here ψ_1 is constant and $\int_0^{2\pi}\frac{1}{\rho}\frac{\delta\psi_2}{\delta u}=0$, also

$$x\psi_2=\tfrac{1}{2}V(\rho^2-\lambda a^2)=\frac{x\psi_0}{a^2}(\rho^2-\lambda a^2)$$

Hence

$$\gamma=\frac{\pi}{a^2\psi_0}\int_0^\pi(\rho^2-\lambda a^2)\frac{1}{\rho}\frac{\delta\psi_1}{du}dv.$$

Further $\int_0^\pi\frac{\mu}{\rho}\frac{\delta\psi_1}{du}dv$ is the flow along a closed curve threading the aperture and is therefore the cyclic constant μ. Therefore

$$\gamma=-\frac{\pi\lambda}{\psi_0}+\frac{\pi}{a^2\psi_0}\int_0^\pi\rho\frac{\delta\psi_1}{du}dv$$

The last integral may be expressed as in the analogous case for β.

Section III.—*Steady motion of hollow vortex.*

9. The form of a hollow vortex and its motion are conditioned by the fact that the velocity of the fluid relatively to the hollow, when the motion is steady, must be constant over the whole surface of the tore. When the section is small compared

with the aperture, the section will clearly be very approximately circular, and to a first approximation the motion will be represented by the stream function found in the previous section, the value of x therein being chosen so as to make the coefficient of $\cos v$ in the expression for the velocity disappear. This will give the first term in the expression for the velocity of translation of the vortex, when it moves forward without change of form. In order to arrive at closer approximations it will be necessary to take account of the form of the section, and this is done in the following investigation, so far as to get a second approximation, although the method employed is capable of being carried further, of course, with more and more complexity in the calculations.

By impressing on the whole fluid a velocity equal and opposite to that of the hollow, the hollow is brought to rest, with the fluid streaming past it. The stream function in this case becomes

$$\psi=-\tfrac{1}{2}a^2\mathrm{V}\frac{\mathrm{S}^2}{(\mathrm{C}-c)^2}+\frac{2\psi_0\sqrt{2}}{\pi\sqrt{(\mathrm{C}-c)}}\Sigma\mathrm{A}_n\frac{\mathrm{R}_n}{\mathrm{R}'_n}\cos nv$$

where

$$\mathrm{A}_0=\{1+(1-\lambda)x\}\mathrm{T}'_0$$

$$\mathrm{A}_n=2\left\{\left(1+\frac{\lambda}{4n^2-1}\right)x-\frac{1}{4n^2-1}\right\}\mathrm{T}'_n$$

The values of the first three are

$$\left.\begin{aligned}
\mathrm{A}_0&=\frac{\pi}{4}[1+\tfrac{1}{4}k^2+\tfrac{1}{64}k^4+\{4(\mathrm{L}-2)k^2+(2\mathrm{L}^2-\tfrac{19}{2}\mathrm{L}+14)k^4\}x]k^{-\frac{1}{2}}\\
\mathrm{A}_1&=\frac{\pi}{4}[\{4-\tfrac{1}{2}(8\mathrm{L}-15)k^2-(2\mathrm{L}^2-11\mathrm{L}+17\tfrac{1}{16})k^4\}xk-(1-\tfrac{1}{8}k^2-\tfrac{1}{64}k^4)k]k^{-\frac{1}{2}}\\
\mathrm{A}_2&=\frac{\pi}{16}[\{16k^2-4(\mathrm{L}-1)k^4\}x-(k^2-\tfrac{1}{4}k^4)]k^{-\frac{1}{2}}
\end{aligned}\right\}\quad(19)$$

The approximation proceeding according to powers of k, each coefficient is one order higher than the preceding.

Let U be the velocity at any point of the hollow. Then, to the first order of small quantities, where the section is circular

$$\mathrm{U}=\left[\frac{1}{\rho}\frac{\delta\psi}{\delta u}\frac{du}{dn}\right]_{u=u'}=\left[\frac{(\mathrm{C}-c)^2}{a^2\mathrm{S}}\frac{\delta\psi}{\delta u}\right]_{u=u'}$$

where

$$\frac{\psi}{2\psi_0}=-\frac{1}{2}\frac{x\mathrm{S}^2}{(\mathrm{C}-c)^2}+\frac{\sqrt{2}}{\pi\sqrt{(\mathrm{C}-c)}}\left(\mathrm{A}_0\frac{\mathrm{R}_0}{\mathrm{R}'_0}+\mathrm{A}_1\frac{\mathrm{R}_1}{\mathrm{R}'_1}\cos v\right)$$

The part of U due to the first term is

$$\frac{\mathrm{U}_1}{2\psi_0}=-\frac{x}{a^2}\cdot\frac{1-\mathrm{C}c}{\mathrm{C}-c}$$

In finding the second part it will be well for the later approximations to carry ψ a term further to include A_2. Then from (7), if U_2 be the part of U due to this

$$\frac{a^2U_2}{2\psi_0}=\frac{\sqrt{2}}{2\pi}\sqrt{(C-c)}\{B_0+\Sigma B_n \cos nv\}$$

where

$$B_0=-\tfrac{1}{2}A_0\frac{P_1}{R'_0}-\tfrac{3}{4}A_1\frac{P_1}{R'_1}$$

$$B_1=\tfrac{1}{2}A_0\frac{P_0}{R'_0}+\tfrac{3}{4}A_1\frac{(P_0+P_2)}{R'_1}-\tfrac{15}{4}A_2\frac{P_2}{R'_2}$$

$$B_n=\left\{(n^2-\tfrac{1}{4})\frac{A_n}{R'_n}-(\overline{n+1}|^2-\tfrac{1}{4})\frac{A_{n+1}}{R'_{n+1}}\right\}P_{n+1}-\left\{(\overline{n-1}|^2-\tfrac{1}{4})\frac{A_{n-1}}{R'_{n-1}}-(n^2-\tfrac{1}{4})\frac{A_n}{R'_n}\right\}P_{n-1}$$

These values of U_1, U_2 are to be expanded in a series of cosines of multiple angles. But here it is only needful to keep terms of the same order as A_2, or compared with A_0 of order k^2. Now

$$2C=k+\frac{1}{k}$$

Hence

$$\frac{1-Cc}{C-c}=\left(\frac{1}{C}-c\right)\left(1+\frac{1}{2C^2}+\frac{\cos v}{C}+\frac{\cos 2v}{2C^2}\right)$$

$$=\frac{1}{2C}-\left(1-\frac{1}{4C^2}\right)\cos v-\frac{\cos 2v}{2C}-\frac{1}{4C^2}\cos 3v$$

$$=k(1-k^2)-(1-k^2)\cos v-k(1-k^2)\cos 2v-k^2\cos 3v$$

Also

$$\sqrt{2}(C-c)^{\frac{1}{2}}\{B_0+\Sigma B_n \cos nv\}$$

$$=\sqrt{2C}\left\{1-\frac{1}{16C^2}-\frac{\cos v}{2C}-\frac{\cos 2v}{16C^2}\right\}\{B_0+B_1\cos v+B_2\cos 2v+B_3\cos 3v\}$$

$$=\frac{1+\frac{1}{2}k^2}{\sqrt{k}}\{1-\tfrac{1}{4}k^2-k(1-k^2)\cos v-\tfrac{1}{4}k^2\cos 2v\}\{B_0+\ldots\}$$

$$=\{1+\tfrac{1}{4}k^2-k\cos v-\tfrac{1}{4}k^2\cos 2v\}\{B_0+B_1\cos v+B_2\cos 2v\}k^{-\frac{1}{2}}$$

$$=\{1+\tfrac{1}{4}k^2-k\cos v-\tfrac{1}{4}k^2\cos 2v\}(B_0+B_1\cos v)k^{-\frac{1}{2}}+(1-k\cos v)B_2k^{-\frac{1}{2}}\cos 2v$$

considering at present B_0 and B_1 to be of the same order. From this it is easy to show that if

$$\left.\begin{aligned}
\frac{a^2\mathrm{U}}{2\psi_0}&=\alpha+\beta\cos v+\gamma\cos 2v+\delta\cos 3v\\
\alpha&=-k(1-k^2)x+\frac{1}{2\pi}(1+\tfrac{1}{4}k^2)\mathrm{B}_0k^{-\frac{1}{2}}-\frac{1}{4\pi}\mathrm{B}_1k^{\frac{1}{2}}\\
\beta&=(1-k^2)x-\frac{1}{2\pi}\mathrm{B}_0k^{\frac{1}{2}}+\frac{1}{2\pi}(1+\tfrac{1}{8}k^2)\mathrm{B}_1k^{-\frac{1}{2}}-\frac{1}{4\pi}\mathrm{B}_2k^{\frac{1}{2}}\\
\gamma&=k(1-k^2)x-\frac{1}{8\pi}\mathrm{B}_0k^{\frac{3}{2}}-\frac{1}{4\pi}\mathrm{B}_1k^{\frac{1}{2}}+\frac{1}{2\pi}\mathrm{B}_2k^{-\frac{1}{2}}\\
\delta&=k^2x-\frac{1}{16\pi}\mathrm{B}_1k^{\frac{3}{2}}-\frac{1}{4\pi}\mathrm{B}_2k^{\frac{1}{2}}+\mathrm{B}_3k^{-\frac{1}{2}}
\end{aligned}\right\} \quad . \quad . \quad . \quad (20)$$

For the first approximation the lowest term in k in β must vanish. Hence

$$x-\frac{1}{2\pi}\mathrm{B}_0k^{\frac{1}{2}}+\frac{1}{2\pi}\mathrm{B}_1k^{-\frac{1}{2}}=0$$

Now

$$\mathrm{B}_0=-\tfrac{1}{2}\mathrm{A}_0\frac{\mathrm{P}_1}{\mathrm{R}_0}-\tfrac{3}{4}\frac{\mathrm{A}_1\mathrm{P}_1}{\mathrm{R}_1}$$

and the lowest terms in B_1 are

$$\mathrm{B}_1=\tfrac{3}{4}\mathrm{A}_1\frac{\mathrm{P}_2}{\mathrm{R}_1}+\tfrac{1}{2}\mathrm{A}_0\frac{\mathrm{P}_0}{\mathrm{R}_0}$$

Substituting the values of A_0, &c., from (19)

$$\frac{1}{2\pi}\mathrm{B}_0=\tfrac{1}{16}k^{-\frac{1}{2}}\frac{2}{\frac{1}{2}\mathrm{L}-1}-\tfrac{3}{4}\frac{4x-1}{8}k^{\frac{1}{2}}4k$$

$$=\tfrac{1}{4}\frac{k^{-\frac{1}{2}}}{\mathrm{L}-2}-\tfrac{3}{8}(4x-1)k^{\frac{3}{2}}$$

$$\frac{1}{2\pi}\mathrm{B}_1=\tfrac{3}{32}k^{\frac{1}{2}}(4x-1)\frac{\frac{4}{3}k^{-\frac{3}{2}}}{\frac{1}{2}k^{-\frac{3}{2}}}-\tfrac{1}{16}k^{-\frac{1}{2}}\frac{2\mathrm{L}k^{\frac{1}{2}}}{(\frac{1}{2}\mathrm{L}-1)k^{-\frac{1}{2}}}$$

$$=\left\{\tfrac{1}{4}(4x-1)-\tfrac{1}{4}\frac{\mathrm{L}}{\mathrm{L}-2}\right\}k^{\frac{1}{2}}$$

Hence

$$x-\frac{1}{4(\mathrm{L}-2)}+x-\tfrac{1}{4}-\tfrac{1}{4}\frac{\mathrm{L}}{\mathrm{L}-2}=0$$

$$2x=\tfrac{1}{4}\frac{2\mathrm{L}-1}{\mathrm{L}-2}$$

$$x=\tfrac{1}{8}\frac{2\mathrm{L}-1}{\mathrm{L}-2}$$

To the same order

$$\frac{2\psi_0}{a^2}=\frac{\mu}{\pi a}(\mathrm{L}-2) \qquad \text{by (15)}$$

Therefore

$$\mathrm{V}=\frac{\mu}{4\pi a}(\mathrm{L}-\tfrac{1}{2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The principal term in U is found by equating it to the principal terms in α, *i.e.*,

$$\frac{a^2\mathrm{U}_0}{2\psi_0}=\frac{1}{2\pi}\,\mathrm{B}_0 k^{-\frac{1}{2}}=\tfrac{1}{4}\,\frac{k^{-1}}{\mathrm{L}-2}$$

and is therefore independent of the velocity of translation, as ought to be the case, since the latter depends on the difference of the cyclic tangential velocities inside and outside the tore. Substituting for ψ_0

$$\mathrm{U}_0=\frac{\mu}{4\pi a k}$$

Now, for steady motion, the equation of pressure gives at the surface of the hollow, if Π and ρ be the pressure at an infinite distance, and the density of the fluid respectively

$$\frac{2\Pi}{\rho}=\mathrm{U}_0^{\,2}$$

Hence U must be the same for hollows of all sizes, and consequently ak constant for all the steady motions of the same vortex. When the hollow is small this is approximately the same as saying that the radius of the cross section is constant. The corresponding theorem for a solid ring is of course that the volume is constant.

10. For the second approximation we need to determine the stream function when the cross section of the ring is not an exact circle. The following investigation is slightly more general than is necessary for our present purposes.

Let k be the value of k for the mean section, and let the section be given by

$$\kappa=k+\Sigma(\mathrm{M}_n\cos nv+\mathrm{N}_n\sin nv)=k+\xi,\ \text{say}$$

where M_n, N_n are small quantities with respect to k. When the tore is at rest with fluid streaming past it, the stream function is

$$\psi=-x\psi_0\frac{\mathrm{S}^2}{(\mathrm{C}-c)^2}+\frac{2\psi_0\sqrt{2}}{\pi\sqrt{(\mathrm{C}-c)}}\,\Sigma\mathrm{A}_n\frac{\mathrm{R}_n}{\mathrm{R}'_n}\cos nv$$

where the A_n have the values given in (19).

Let the stream function for the non-circular section be $\psi+\chi$ where

$$\chi=\frac{1}{\surd(\mathrm{C}-c)}\Sigma(\mathrm{X}_n\cos nv+\mathrm{Y}_n\sin nv)\frac{\mathrm{R}_n}{\mathrm{R}'_n}$$

and X_n, Y_n are also small. The necessary condition is, that when κ has the value given above, $\psi+\chi$ must be constant, say $\psi'_0+\epsilon$. Then neglecting squares of ξ

$$\psi'_0+\epsilon=\psi'_0+\left.\frac{\delta\psi}{\delta\kappa}\right|\xi+\frac{1}{\surd(\mathrm{C}_0-c)}\Sigma(\mathrm{X}_n\cos nv+\mathrm{Y}_n\sin nv)$$

The value of ϵ is arbitrary, since with any given surface conditions the circulation remains undetermined. We shall choose it so as to make the circulation zero. It would be impossible to determine X_n, Y_n in the general case where both ξ and $\delta\psi/\delta\kappa$ are infinite series; but in the case required in the present paper $\mathrm{A}_n/\mathrm{A}_{n-1}$ is of the order k, k being small, and the terms in A_n are neglected after A_2. This simplifies the calculation, and it is easy to determine the terms X_n, Y_n in terms of M_n, N_n. But it is further greatly simplified by the fact that in the case to which we have to apply it the velocity along the surface is already uniform to the first order—in other words

$$\mathrm{U}=\frac{(\mathrm{C}_0-c)^2}{a^2\mathrm{S}_0}\left(\frac{\delta\psi}{\delta\kappa}\right)_0\frac{d\kappa}{du}$$

whence

$$\left(\frac{\delta\psi}{\delta\kappa}\right)_0=-\frac{a^2\mathrm{S}_0}{k(\mathrm{C}_0-c)^2}\mathrm{U}$$

Hence the equation determining the X_n, Y_n is

$$\epsilon=-\frac{a^2\mathrm{S}_0}{k(\mathrm{C}_0-c)^2}\mathrm{U}\xi+\frac{1}{\surd(\mathrm{C}_0-c}\Sigma(\mathrm{X}_n\cos nv+\mathrm{Y}_n\sin nv)$$

But since in our applications k is itself so small that k^3 has been neglected compared with unity, the above becomes

$$\begin{aligned}\epsilon=&-2a^2(1+3k^2+4k\cos v+6k^2\cos 2v)\mathrm{U}\xi\\&+\surd(2k)(1+\tfrac{1}{4}k^2+k\cos v+\tfrac{3}{4}k^2\cos 2v)\Sigma(\mathrm{X}_n\cos nv+\mathrm{Y}_n\sin nv)\end{aligned}$$

The various normal functions will therefore be composed of a set of principal terms in cos nv, &c., each corrected by an infinite convergent series of small terms of the others. The principal will be given by

$$\epsilon=-2a^2\mathrm{U}\Sigma_1(\mathrm{M}_n\cos nv+\mathrm{N}_n\sin nv)+\surd(2k)\Sigma(\mathrm{X}_n\cos nv+\mathrm{Y}_n\sin nv)$$

Therefore

$$\mathrm{X}_0=\frac{\epsilon}{\surd(2k)},\ \mathrm{Y}_0=0$$

$$\mathrm{X}_n=\frac{2a^2\mathrm{U}}{\surd(2k)}\mathrm{M}_n,\ \mathrm{Y}_n=\frac{2a^2\mathrm{U}}{\surd(2k)}\mathrm{N}_n$$

The series connected with X_n, Y_n to complete the solution for a given M_n, N_n are found from

$$\begin{aligned}0=&-2a^2Uk(3k+4\cos v+6k\cos 2v)M_n\cos nv\\&+k(\tfrac{1}{4}k+\cos v+\tfrac{3}{4}k\cos 2v)(\epsilon+2a^2UM_n\cos nv)\\&+\sqrt{(2k)}(1+k\cos v)\Sigma X_n\cos nv\end{aligned}$$

with a corresponding equation for Y_n in which $\epsilon=0$.

We need only consider for the first approximation the principal terms, which give

$$\chi\sqrt{(C-c)}=\frac{\epsilon}{\sqrt{(2k)}}\cdot\frac{R_0}{R'_0}+\frac{2a^2U}{\sqrt{(2k)}}(M_n\cos nv+N_n\sin nv)\frac{R_n}{R'_n}$$

Since the circulation is to vanish,

$$\epsilon+2a^2M_nU=0$$

$$\chi=\frac{2a^2U}{\sqrt{2k}}\frac{1}{\sqrt{(C-c)}}\left\{-M_n\frac{R_0}{R'_0}+(M_n\cos nv+N_n\sin nv)\frac{R_n}{R'_n}\right\}$$

11. We are now in a position to determine the first term in the expression, giving the form of the hollow, viz., that part which will destroy the term in cos $2v$ in the value of the surface velocity. $\overline{U}$ denoting this velocity we have .

$$\overline{U}^2=\frac{(C-c)^4}{a^4S^2}\left\{\left(\frac{\delta\psi}{\delta u}+\frac{\delta\chi}{\delta u}\right)^2+\left(\frac{\delta\psi}{\delta v}+\frac{\delta\chi}{\delta v}\right)^2\right\}$$

Now at the mean section $\delta\psi/\delta v=0$, and is therefore at least of the first order of small quantities near the circle $u=u_0$. Hence

$$\begin{aligned}\overline{U}&=\frac{(C-c)^2}{a^2S}\left\{\frac{\delta\psi}{\delta u}+\frac{\delta\chi}{\delta u}\right\}\\&=U_1+\frac{(C_0-c)^2}{a^2S_0}\left(\frac{d\chi}{\delta u}\right)_0\end{aligned}$$

where

$$\frac{a^2U_1}{2\psi_0}=\alpha+\beta\cos v+\gamma\cos 2v$$

and α, β, γ are the values given in (20) when $k+\xi$ is substituted for k *in the functions in* u.

Now ξ is of the form M cos $2v$, hence

$$\chi=\frac{2a^2U_0}{\sqrt{(2k)}}\cdot\frac{M}{\sqrt{(C-c)}}\left\{-\frac{R_0}{R'_0}+\frac{R_2}{R'_2}\cos 2v\right\}$$

Therefore

$$\left(\frac{\delta\chi}{\delta u}\right)_0=\frac{2a^2\mathrm{MU}_0}{\sqrt{(2k)}}\left\{-\tfrac{1}{2}\frac{\mathrm{S}}{(\mathrm{C}-c)^{\frac{3}{2}}}(-1+\cos 2v)+\frac{\mathrm{S}}{(\mathrm{C}-c)^{\frac{1}{2}}}\left(\tfrac{1}{4}\frac{\mathrm{P}_0}{\mathrm{R}_0}+\tfrac{15}{4}\frac{\mathrm{P}_2}{\mathrm{R}_2}\cos 2v\right)\right\}$$

and

$$\frac{(\mathrm{C}-c)^2}{a^2\mathrm{S}}\left(\frac{\delta\chi}{\delta u}\right)_0=\frac{(\mathrm{C}-c)^{\frac{1}{2}}\mathrm{MU}_0}{2\sqrt{(2k)}}\left\{2-2\cos 2v+(\mathrm{C}-c)\left(\frac{\mathrm{P}_0}{\mathrm{R}_0}+15\frac{\mathrm{P}_2}{\mathrm{R}_2}\cos 2v\right)\right\}$$

The principal term here is

$$\frac{\mathrm{MU}_0}{4k}\left(2-2\cos v+\frac{\mathrm{CP}_0}{\mathrm{R}_0}+15\frac{\mathrm{CP}_2}{\mathrm{R}_2}\cos 2v\right)$$

$$=\frac{\mathrm{MU}_0}{4k}\left(2-\frac{2\mathrm{L}}{\mathrm{L}-2}-2\cos 2v+10\cos 2v\right)$$

$$=-\frac{\mathrm{MU}_0}{k}\left(\frac{1}{\mathrm{L}-2}-2\cos 2v\right)$$

It remains now to determine α, β, γ to the same order of approximation, that is so far as the first power of k,

$$\alpha=-\kappa x+\frac{1}{2\pi}\mathrm{B}_0\kappa^{-\frac{1}{2}}-\frac{1}{4\pi}\mathrm{B}_1\kappa^{\frac{1}{2}}$$

$$=-kx+\frac{1}{2\pi}\left[\mathrm{B}_0k^{-\frac{1}{2}}-\tfrac{1}{2}\mathrm{B}_1k^{\frac{1}{2}}\right]_0-\xi x_0-\frac{1}{2\pi}\left[\frac{d}{du}(\mathrm{B}_0\kappa^{-\frac{1}{2}})\right]_0\frac{\xi}{k}$$

$$=-k(x_0+\delta x)+\tfrac{1}{4}\frac{k^{-1}}{\mathrm{L}-2}-\tfrac{3}{8}(4x_0-1)k-\tfrac{1}{8}\left(4x_0-1-\frac{\mathrm{L}}{\mathrm{L}-2}\right)k-\left\{x_0k+\frac{1}{2\pi}\frac{d}{du}(\mathrm{B}_0\kappa^{-\frac{1}{2}})\right\}_0\frac{\xi}{k}$$

$$=-kx_0+\tfrac{1}{4}\frac{k^{-1}}{\mathrm{L}-2}-\tfrac{1}{2}\left(4x_0-1-\tfrac{1}{4}\frac{\mathrm{L}}{\mathrm{L}-2}\right)k-\left\{x_0k+\frac{1}{2\pi}\frac{d}{du}(\mathrm{B}_0\kappa^{-\frac{1}{2}})\right\}\frac{\xi}{k}$$

Now

$$\mathrm{B}_0=-\tfrac{1}{2}\mathrm{A}_0\frac{\mathrm{P}_1}{\mathrm{R}'_0}-\tfrac{3}{4}\mathrm{A}_1\frac{\mathrm{P}_1}{\mathrm{R}'_1}$$

therefore

$$\frac{d}{du}(\mathrm{B}_0\kappa^{-\frac{1}{2}})=\tfrac{1}{2}\mathrm{B}_0k^{-\frac{1}{2}}-k^{-\frac{1}{2}}\left\{\tfrac{1}{2}\mathrm{A}_0\frac{\frac{d\mathrm{P}_1}{du}}{\mathrm{R}'_0}+\tfrac{3}{4}\mathrm{A}_1\frac{\frac{d\mathrm{P}_1}{du}}{\mathrm{R}'_1}\right\}$$

therefore

$$\frac{1}{2\pi}\frac{d}{du}(\mathrm{B}_0\kappa^{-\frac{1}{2}})_0=\left\{\tfrac{1}{4}\frac{k^{-1}}{\mathrm{L}-2}-\tfrac{3}{8}(4x_0-1)k\right\}\left\{\tfrac{1}{2}+\frac{\frac{d\mathrm{P}_1}{du}}{\mathrm{P}_1}\right\}$$

$$=\tfrac{1}{4}\left\{\frac{k^{-1}}{\mathrm{L}-2}-\tfrac{3}{2}(4x_0-1)k\right\}$$

Therefore, substituting the value of x_0 already found

$$\alpha=-\tfrac{1}{2}\left(6x_0-1-\tfrac{1}{4}\frac{L}{L-2}\right)k+\tfrac{1}{4}\frac{k^{-1}}{L-2}-\tfrac{1}{4}\left\{\frac{k^{-1}}{L-2}-(2x_0-\tfrac{3}{2})k\right\}\frac{\xi}{\kappa}$$

$$=-\tfrac{1}{8}\frac{L+5}{L-2}k+\tfrac{1}{4}\frac{k^{-1}}{L-2}-\tfrac{1}{4}\left(\frac{k^{-1}}{L-2}\right)\frac{\xi}{k}$$

for, since ξ is at least of order k^2, the last term in the factor of ξ, can be neglected, and

$$\alpha=-\tfrac{1}{8}\frac{L+5}{L-2}k+\tfrac{1}{4}\frac{k^{-1}}{L-2}-\tfrac{1}{4}\frac{k^{-2}}{L-2}\xi$$

again

$$\beta=\beta_0+\delta x_0+\frac{1}{2\pi}\left[\frac{d}{du}(B_0\kappa^{\frac{1}{2}}-B_1\kappa^{-\frac{1}{2}})\right]_0\frac{\xi}{k}-\frac{1}{2\pi}\frac{d}{dx}(B_0k^{\frac{1}{2}}-B_1k^{-\frac{1}{2}})\delta x_0$$

Now

$$\frac{1}{2\pi}\frac{d}{du}(B_0\kappa^{\frac{1}{2}})=\frac{1}{2\pi}\left\{\kappa\frac{d}{du}(B_0\kappa^{-\frac{1}{2}})+B_0\kappa^{-\frac{1}{2}}\frac{d\kappa}{du}\right\}$$

$$=-\tfrac{1}{4}\frac{1}{L-2}+\tfrac{1}{4}\frac{1}{L-2}=0$$

Also since

$$B_1=\tfrac{3}{4}A_1\frac{P_2}{R'_1}+\tfrac{1}{2}A_0\frac{P_0}{R'_0}$$

$$\frac{dB_1}{du}=\tfrac{3}{4}A_1\frac{R_2}{SR_1}+\tfrac{1}{2}A_0\frac{1}{S}$$

$$\frac{1}{2\pi}\frac{dB_1}{du}=\tfrac{3}{8}(4x-1)k^{\frac{1}{2}}+\tfrac{1}{8}k^{\frac{1}{2}}$$

$$\frac{1}{2\pi}\frac{d}{du}(B_1\kappa^{-\frac{1}{2}})=\tfrac{3}{8}(4x-1)+\tfrac{1}{8}+\tfrac{1}{2}\left\{\tfrac{1}{4}(4x-1)-\tfrac{1}{4}\cdot\frac{L}{L-2}\right\}$$

$$=\tfrac{1}{2}(4x_0-1)-\tfrac{1}{4}\frac{1}{L-2}=\tfrac{1}{2}\frac{1}{L-2}$$

also

$$\frac{1}{2\pi}\frac{d}{dx}(B_0k^{\frac{1}{2}}-B_1k^{-\frac{1}{2}})=-1$$

therefore

$$\beta=2\delta x_0-\tfrac{1}{2}\frac{1}{L-2}\frac{\xi}{k}$$

$$\gamma=x_0k-\frac{1}{8\pi}B_0k^{\frac{3}{2}}-\frac{1}{4\pi}B_1k^{\frac{1}{2}}+\frac{1}{2\pi}B_2k^{-\frac{1}{2}}$$

The principal term in B_2 is

$$B_2=\tfrac{15}{4}A_2\frac{P_3}{R_2}-\tfrac{3}{4}A_1\frac{P_1}{R_1}$$

therefore

$$\frac{1}{2\pi}\mathrm{B}_2=\tfrac{15}{4}\cdot\tfrac{1}{32}(16x-1)k^{\frac{3}{2}}\frac{\frac{16}{15}k^{-\frac{5}{2}}}{k^{-5}}-\tfrac{3}{4}\cdot\tfrac{1}{8}(4x-1)k^{\frac{1}{2}}\frac{2k^{-\frac{1}{2}}}{\frac{1}{2}k^{-\frac{3}{2}}}$$

$$=\tfrac{1}{4}(2x+1)k^{\frac{3}{2}}$$

therefore

$$\gamma=\left\{x_0-\tfrac{1}{16}\cdot\frac{1}{\mathrm{L}-2}-\tfrac{1}{8}\left(4x_0-1-\frac{\mathrm{L}}{\mathrm{L}-2}\right)+\tfrac{1}{4}(2x_0+1)\right\}k$$

$$=\left(x_0+\tfrac{3}{8}+\tfrac{1}{16}\frac{2\mathrm{L}-1}{\mathrm{L}-2}\right)k=\tfrac{1}{16}\frac{12\mathrm{L}-15}{\mathrm{L}-2}k$$

Hence, substituting their values for α, β, γ

$$\overline{\mathrm{U}}=\left\{\tfrac{1}{4}\frac{k^{-1}}{\mathrm{L}-2}-\tfrac{1}{8}\frac{\mathrm{L}+5}{\mathrm{L}-2}k-\tfrac{1}{4}\cdot\frac{k^{-2}}{\mathrm{L}-2}\mathrm{M}\cos 2v\right\}\frac{2\psi_0}{a^2}$$

$$+\left\{\left(2\delta x_0-\tfrac{1}{2}\frac{1}{\mathrm{L}-2}\frac{\mathrm{M}}{k}\cos 2v\right)\frac{2\psi_0}{a^2}\right.$$

$$\left.+\frac{2\psi_0}{a^2}\cdot\tfrac{1}{16}\frac{12\mathrm{L}-15}{\mathrm{L}-2}k\cos 2v-\frac{\mathrm{MU}_0}{k}\left(\frac{1}{\mathrm{L}-2}-2\cos 2v\right)\right.$$

The condition that the coefficient of $\cos 2v$ vanishes gives

$$\mathrm{M}\left\{\frac{2\mathrm{U}_0}{k}-\tfrac{1}{4}\frac{2\psi_0}{a^2k^2(\mathrm{L}-2)}\right\}+\tfrac{1}{16}\cdot\frac{12\mathrm{L}-15}{\mathrm{L}-2}k\cdot\frac{2\psi_0}{a^2}=0$$

or since

$$\mathrm{U}_0=\tfrac{1}{4}\frac{k^{-1}}{\mathrm{L}-2}\cdot\frac{2\psi_0}{a^2}$$

therefore

$$\tfrac{1}{4}\frac{\mathrm{M}k^{-2}}{\mathrm{L}-2}=-\tfrac{1}{16}\frac{12\mathrm{L}-15}{\mathrm{L}-2}k$$

or

$$\mathrm{M}=-\tfrac{1}{4}(12\mathrm{L}-15)k^3$$

Also, since $\beta=0$,

$$\delta x_0=0$$

Therefore

$$\overline{\mathrm{U}}=\frac{2\psi_0}{a^2}\left\{\tfrac{1}{4}\frac{k^{-1}}{\mathrm{L}-2}-\tfrac{1}{8}\frac{\mathrm{L}+5}{\mathrm{L}-2}k\right\}$$

$$=\frac{\mu}{4\pi ak}\{1-\tfrac{1}{2}(\mathrm{L}+5)k^2\}$$

Hence,

(1.) The velocity of translation remains unaltered to this order.

(2.) The form of the hollow is given by

$$\kappa=k\{1-\tfrac{1}{4}(12\mathrm{L}-15)k^2\cos 2v\}$$

(3.) The surface velocity is

$$\frac{\mu}{4\pi ak}\{1-\tfrac{1}{2}(\mathrm{L}+5)k^2\}$$

The effect of the correction to the form of the hollow is to make the section slightly elliptic with the major axis perpendicular to the plane of the ring, and with the inner side slightly flatter than the other.

The value of x obtained above is, when k is infinitely small, larger than the critical value given in (18). The fluid carried forward will therefore be ring-shaped. If for a rough approximation we take the two first terms of the expressions, the value of k for a hollow vortex in steady motion, and carrying forward a simply connected mass of fluid, will be found from

$$L=\frac{4\pi+1}{2}$$

or

$$k=4e^{-\frac{4\pi+1}{2}}=4e^{-6\cdot 78}$$

$$=\cdot 00522 \text{ very nearly}$$

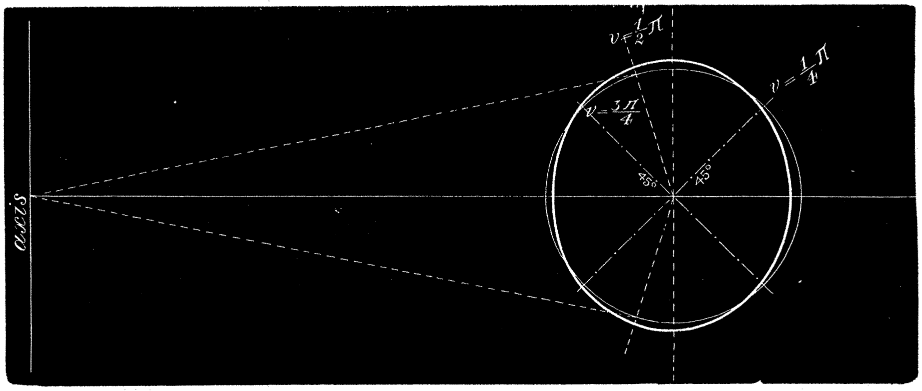
this would make $R/r=10^2$, about. Since k is so small our approximations are very close, and it follows that for even extremely small cores of hollow vortices, the fluid carried forward is a mass without aperture. For infinitely small ones this is not the case.

The form of the hollow has been determined above by the value of k. If the normal variation from the circle be denoted by δn

$$\delta n=\frac{dn}{du}\frac{du}{dk}\delta k=-\tfrac{1}{4}\frac{a}{C-c}(12L-15)k^2\cos 2v$$

$$=-\tfrac{1}{16}\frac{r^3}{R^2}\left(12\log_e\frac{8R}{r}-15\right)\cos 2v$$

The figure below represents the form given by this expression to a circle in which $r/R=\cdot 2$. Though this value is large, it shows the nature of the change of form better than a smaller value.



12. On account of the constant surface velocity along a hollow vortex, with fluid streaming past it, it is very easy to determine the energy of the motion, when the ring moves through a fluid otherwise at rest. For § 8 the energy is given by

$$\pi\int_0^{2\pi}\frac{\psi}{\rho}\frac{\delta\psi}{\delta u}dv=\pi\int_0^l \mathrm{U}'\psi ds$$

where ds is an element of the arc of the cross-section, l its length, and U′ the velocity along the surface regarded as for the moment fixed in space. It is therefore the component of V along the surface and U, that is

$$\mathrm{U}'=\mathrm{U}+\mathrm{V}\frac{\mathrm{R}-\rho}{r}$$

Also

$$\psi=\tfrac{1}{2}\mathrm{V}\rho^2+\text{ const}$$
$$=\tfrac{1}{2}\mathrm{V}(\rho^2-\lambda a^2)+\psi_0$$

Hence, to the first order, where the section is a circle, the energy is

$$\pi\int_0^{2\pi}\left(\mathrm{U}+\mathrm{V}\frac{\mathrm{R}-\rho}{r}\right)\left(\psi_0-\frac{\mathrm{V}}{2}\lambda a^2+\tfrac{1}{2}\mathrm{V}\rho^2\right)rd\theta$$

where

$$\rho=\mathrm{R}+r\cos\theta$$

Therefore the energy is

$$\pi r\int_0^{2\pi}[\mathrm{U}\{\psi_0+\tfrac{1}{2}\mathrm{V}(\mathrm{R}^2-\lambda a^2)\}+(\tfrac{1}{2}\mathrm{UV}r^2+\mathrm{V}^2\mathrm{R}r)\cos^2\theta]d\theta$$
$$=2\pi^2r[\mathrm{U}\{\psi_0+\tfrac{1}{2}\mathrm{V}(\mathrm{R}^2-\lambda a^2)\}+\tfrac{1}{2}\mathrm{V}r(\tfrac{1}{2}\mathrm{U}r+\mathrm{VR})]$$

But

$$\mathrm{U}=\frac{\mu}{4\pi ak}$$

$$\mathrm{V}=\frac{\mu}{4\pi a}(\mathrm{L}-\tfrac{1}{2})$$

$$\psi_0=\frac{\mu a}{2\pi}(\mathrm{L}-2)$$

Substituting these values, the energy is

$$\tfrac{1}{4}\mu^2rk^{-1}\left\{\mathrm{L}-2+\frac{1}{4a^2}(\mathrm{R}^2+\tfrac{1}{2}r^2-\lambda a^2)(\mathrm{L}-\tfrac{1}{2})+\frac{\mathrm{R}r}{4a^2}(\mathrm{L}-\tfrac{1}{2})^2k\right\}$$

Now to the order of approximation of circular section $\lambda=1$, $r=2ak$, $\mathrm{R}=a$, and the energy is $\frac{1}{2}\mu^2a(\mathrm{L}-2)$, which is the same as for a rigid tore at rest. If the shape be regarded, then since here the variation from the circle depends on k^3, we may treat it as circular in the integration, provided we do not carry our approximation beyond k^2. In this case

$$\lambda=1-4(L-2)k^2, \quad r=2ak, \quad R=a(1-2k^2)$$

and the energy is

$$\tfrac{1}{2}\mu^2a\{L-2+(L-\tfrac{5}{2})(L-\tfrac{1}{2})k^2+\tfrac{1}{2}(L-\tfrac{1}{2})^2k^2\}$$
$$=\tfrac{1}{2}\mu^2a\{(L-2)+\tfrac{1}{4}(2L-1)(3L-11)k^2\}$$

To the lowest order this is

$$=\frac{\mu^3}{4\pi V}(L-2)(L-\tfrac{1}{2})$$

13. If the steady shape as just found receive a slight disturbance symmetrical about the straight axis, a series of waves will be propagated round the hollow. To prove this, and to find the time of oscillation for different modes, will be the aim of the remainder of this paper; and firstly I consider the case where the cross-section is crimped into a form given by $\xi=\delta k=M\cos nv+N\sin nv$, where M.N are small compared with k, and functions of the time. Since they are functions of the time, the volume of the hollow will change, and consequently the stream function will be cyclic. The rate of change of volume is

$$\int_0^{2\pi}\dot{\xi}\frac{dn}{dk}\delta v\frac{dn'}{dv}=\frac{a^2}{k}\int_0^{2\pi}\frac{\dot{M}\cos nv+\dot{N}\sin nv}{(C-c)^2}dv$$

$$=\frac{a^2\dot{M}}{k}\int_0^{2\pi}\frac{\cos nv}{(C-c)^2}dv=-\frac{a^2\dot{M}}{kS}\frac{d}{du}\int_0^{2\pi}\frac{\cos nv}{C-c}dv$$

$$=-\frac{\pi a^2\dot{M}}{kS}\frac{dB_n}{du}$$

where B_n is the coefficient of $\cos nv$ in the expansion of $(C-c)^{-1}$

Hence

$$B_n=\frac{2e^{-nu}}{S}$$

and the rate of change of volume

$$=\frac{2\pi a^2\dot{M}}{kS^2}\left(n+\frac{C}{S}\right)e^{-nu}$$

which is of the order $8\pi a^2\dot{M}(n+1)k^{n+1}$, a quantity beyond that which we neglect. Hence we may employ the stream function. Let then χ denote this function for the small motion given by ξ. The condition to find it is that for all values of the time t

$$\dot{\xi}\frac{dn}{dk}=-\frac{1}{\rho}\frac{d\chi}{dv}\frac{dv}{dn'}$$

Considering first the term $\xi = \dot{M}\cos nv$, the corresponding form for χ will be

$$\chi = \frac{1}{\sqrt{(C-c)}} \Sigma Y_m \frac{R_m}{R'_m} \sin mv$$

the coefficient Y being determined from the condition

$$\dot{M}\cos nv = -\frac{k(C-c)^3}{a^3 S} \frac{d}{dv}\left\{\frac{1}{\sqrt{(C-c)}} \Sigma Y_n \frac{R_m}{R'_m} \sin mv\right\}$$

for all values of v when $u=u'$. Therefore

$$\dot{M}\cos nv = -\frac{k(C-c)^3}{a^3 S} \Sigma \left\{\frac{m\cos mv}{\sqrt{(C-c)}} - \frac{1}{2}\frac{\sin v \sin mv}{(C-c)^{\frac{3}{2}}}\right\} Y_m$$

or

$$2\frac{\dot{M}a^3 S}{k} \cdot \frac{\cos nv}{(C-c)^{\frac{5}{2}}} = -\Sigma\{2mC\cos mv - 2m\cos mv\cos v - \sin mv \sin v\} Y_m$$

$$\frac{2\dot{M}a^3 S}{kC} \cdot \frac{\cos nv}{(C-c)^{\frac{5}{2}}} = -\Sigma\left\{2m\cos mv - \frac{2m-1}{2C}\cos \overline{(m+1)}v - \frac{2m+1}{2C}\cos (m-1)v\right\} Y_m$$

From this we may obtain sequence equations to determine the Y_n; but we require only the most important terms, hence

$$2nY_n = -\frac{2\dot{M}a^3 S}{kC^{\frac{5}{2}}}$$

$$Y_n = -\frac{2\sqrt{2}}{n} a^3 \dot{M} k^{\frac{1}{2}}$$

and

$$\chi = -\frac{2\sqrt{2}a^3 \dot{M} k^{\frac{1}{2}}}{n\sqrt{(C-c)}} \cdot \frac{R_n}{R'_n} \sin nv$$

Since the cyclic motion due to this is zero, there is no correction to be introduced for it as in former cases.

If ϕ be the velocity potential, the condition for a free surface gives

$$0 = \frac{\Pi}{\rho} - \dot{\phi} - \frac{1}{2}(\text{vel})^2 + f(t)$$

$f(t)$ being an arbitrary function of the time. The velocity normal to the surface is of the first order of small quantities, and its square is to be neglected.

The velocity along the surface is

$$U_0 + \left(\frac{dU}{dk}\right)_0 \xi$$

where U_0 is the velocity determined in § 10 and

$$\frac{\Pi}{\rho} - \frac{1}{2}U_0^2 = 0$$

Hence

$$\dot{\phi}+U_0\left(\frac{dU}{dk}\right)_0\xi+f(t)=0$$

Now ϕ is the flow along any curve from a fixed point up to the point in question. Let us take the curve to be formed by a straight line from the centre in the plane of the ring ($v=\pi$) up to the surface ($u=u'$), and then along the ring to the point (u', v). The first part will be a function of the time alone, and will therefore disappear with $f(t)$; of the part along the ring, that due to the cyclic motion will be constant, and therefore the corresponding part in $\dot{\phi}$ will disappear. The part depending on the velocity of translation will be proportional to x, which will introduce a quantity proportional to $\dot{x}$ in $\dot{\phi}$. This will contain terms in $\cos v$, which will not enter again. Hence $\dot{x}$ must be equated to zero, or the velocity of translation will not be affected. There remains only the part depending on the flow along the surface due to the motion χ. This we proceed to find. Denoting it by $\phi_{\prime}$

$$\phi_{\prime}=\int_v^\pi\left[\frac{1}{\rho}\frac{d\chi}{du}\frac{du}{dn}\cdot\frac{dn'}{dv}\right]_{u'}dv$$

$$=-\frac{2a^2\dot{M}\sqrt{(2k)}}{nSR'_n}\int_v^\pi(C-c)\left[\frac{d}{du}\frac{R_n}{\sqrt{(C-c)}}\right]_{u'}\sin nvdv$$

$$=-\frac{2a^2\dot{M}\sqrt{(2k)}}{nS}\int_v^\pi\left\{-\tfrac{1}{2}\frac{S}{\sqrt{(C-c)}}+(n^2-\tfrac{1}{4})\sqrt{(C-c)}\frac{SP_n}{R_n}\right\}\sin nvdv$$

the principal part of which is

$$\phi_{\prime}=-\frac{4a^2\dot{M}k}{n^2}\left(-\tfrac{1}{2}+(n^2-\tfrac{1}{4})\frac{CP_n}{R_n}\right)\{\cos nv-(-1)^n\}$$

The part of this, independent of $\cos nv$, will disappear with $f(t)$.

Further, since $U\frac{dU}{dk}$ is multiplied by ξ, we must only take their lowest terms, which are independent of v. Finally then equating to zero the coefficient of $\cos nv$

$$-\frac{4a^2k}{n^2}\left\{-\tfrac{1}{2}+(n^2-\tfrac{1}{4})\frac{CP_n}{R_n}\right\}\ddot{M}+U_0\frac{dU_0}{dk}\cdot M=0$$

Now

$$U=\frac{2\psi_0}{a^2}\alpha$$

therefore

$$\frac{dU}{dk}=\frac{2\psi_0}{a^2}\frac{d\alpha}{dk}$$

$$=\frac{U}{\alpha}\frac{d\alpha}{dk}$$

To the order here reached

$$\alpha=\frac{1}{2\pi}\left(-\tfrac{1}{2}A_0\frac{P_1 k^{-\frac{1}{2}}}{R'_0}\right)$$

$$\frac{d\alpha}{dk}=\frac{1}{2\pi}\frac{A_0}{R_0}k^{-2}=-\alpha k^{-1}$$

therefore

$$\frac{dU}{dk}=-Uk^{-1}$$

Hence

$$\frac{a^2k^2}{n^2}\left\{(4n^2-1)\frac{CP_n}{R_n}-2\right\}\ddot{M}+U^2M=0$$

Now

$$(4n^2-1)\frac{CP_n}{R_n}-2=2\left(\frac{4nCP_n}{P_{n+1}-P_{n-1}}-1\right)$$

$$=4n\cdot\frac{P_{n+1}+P_{n-1}}{P_{n+1}-P_{n-1}}$$

Therefore

$$\ddot{M}+\frac{nU^2}{4a^2k^2}\cdot\frac{P_{n+1}-P_{n-1}}{P_{n+1}+P_{n-1}}M=0$$

The coefficient of M is always positive; hence the hollow is stable for displacements of this kind, and the time of vibration for displacement of order n is

$$\frac{4\pi ak}{U}\sqrt{\left\{\frac{1}{n}\cdot\frac{P_{n+1}+P_{n-1}}{P_{n+1}-P_{n-1}}\right\}}$$

Since throughout our approximations we have neglected k^2 compared with unity, we may simplify this further by obtaining the value of the expression under the square root to the same order,

Now

$$\frac{(2n+1)(P_{n+1}+P_{n-1})}{(2n+1)(P_{n+1}-P_{n-1})}=\frac{4nCP_n-2P_{n-1}}{4nCP_n-4nP_{n-1}}$$

$$=\frac{1-\dfrac{1}{2nC}\cdot\dfrac{(2n-1)P_{n-1}}{(2n-1)P_n}}{1-\dfrac{1}{C}\cdot\dfrac{(2n-1)P_{n-1}}{(2n-1)P_n}}$$

$$=\frac{1-\dfrac{2n-1}{8n(n-1)C^2}}{1-\dfrac{2n-1}{4(n-1)C^2}}$$

$$=\frac{1-\dfrac{2n-1}{2n(n-1)}k^2}{1-\dfrac{2n-1}{n-1}k^2}$$

$$=1+\frac{(2n-1)^2}{2n(n-1)}k^2$$

The time of vibration may also be written in the forms

$$\frac{\mu\rho}{2\Pi}\sqrt{\frac{1}{n}\left(\frac{P_{n+1}+P_{n-1}}{P_{n+1}-P_{n-1}}\right)} \text{ or } \frac{\mu\rho}{2\Pi\sqrt{n}}$$

which shows that the time is independent of the velocity of translation, a result which has important bearing on the theory that atoms of matter are hollow vortices. For the different orders of vibration, the time of vibration varies inversely as the square root of the number of crests running round the hollow.

14. *Pulsation of hollow.*—In the preceding case, $n=0$ would correspond to pulsations of the hollow, in which therefore the whole motion is a change of volume, and the use of the stream function is not allowable. But as it happens, the application of the velocity potential is here very easy. Let, as in Art. 13, the displacement be given by

$$\delta k=\xi=\left(\frac{C-c}{S}\right)^{\frac{3}{2}}M$$

Then the velocity potential is

$$\phi=\sqrt{C-c}\Sigma A_n\frac{P_n}{P'_n}\cos nv$$

with

$$\frac{\dot{\xi}}{k}\cdot\frac{a}{C-c}=-\frac{C-c}{a}\frac{\delta\phi}{\delta u} \text{ when } u=u'$$

Therefore

$$\frac{a}{k}\frac{\sqrt{(C-c)}}{S^{\frac{3}{2}}}\dot{M}=-\frac{\sqrt{(C-c)}}{a}\Sigma\frac{A_n}{P_n}\left\{\tfrac{1}{2}SP_n+(C-c)\frac{dP_n}{du}\right\}\cos nv$$

whence the principal term is

$$A_0=-\frac{2a^2}{kS^{\frac{3}{2}}}\frac{P_0}{SP_0+2C\dfrac{dP_0}{du}}\dot{M}$$

$$=-2a^2\sqrt{2}k^{-\frac{1}{2}}\frac{4P_0}{k^{-2}P_0+4k^{-1}R_0}\dot{M}$$

$$=-4a^2\sqrt{2}k^{\frac{3}{2}}L\dot{M}$$

Hence

$$\phi=-4a^2\sqrt{2}k^{\frac{3}{2}}L\dot{M}\sqrt{(C-c)}\frac{P_0}{P'_0}$$

as before

$$\dot{\phi}+U\frac{dU}{dk}\xi=0$$

Therefore

$$4a^2\sqrt{(2C)}k^{\frac{3}{2}}L\ddot{M}+U^2k^{-1}M=0$$

$$\ddot{M}+\frac{U^2}{4a^2k^2L}M=0$$

Therefore time of pulsation

$$=\frac{4\pi ak}{U}\sqrt{L}=\frac{\mu\rho}{2\Pi}\left(\log\frac{4}{k}\right)^{\frac{1}{2}}$$

and therefore varies slowly with the energy.